Exercise sheet N. 11
Statistical Physics

University of Heidelberg

`http://www.thphys.uni-heidelberg.de/~amendola/teaching.html`

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Indicate your name, your UNI-ID and your exercise group on each page of the answer sheets.

Please hand in your solution on Tuesday 21th January during the lecture.

1 Condensation for general dispersion [10pt]

An ideal Bose gas whose single particle spectrum is given by $\varepsilon(\vec{p}) = c|p|^\sigma$ with $\sigma > 0$ is contained in volume V of dimension d. The gas is in uniform temperature T.

1. Calculate the one-particle density of states.
2. Find the condition on σ and d for the existence of Bose-Einstein condensation.
3. Find the dependence of the number of particles N on the chemical potential μ.
4. Find the dependence of the total energy E on the chemical potential, and show how the pressure P is obtained from this result.
5. Find an expression for the heat capacity C_v as a function of N in the limit of infinite temperature.

2 Ultra-relativistic Fermi Gas [10pt]

An ultra-relativistic ideal Fermi fluid is contained in a volume V. The chemical potential is μ. Consider only the special case $T = 0$ in the following. Calculate the mean particle number $\langle N \rangle$ and the mean total energy $\langle E \rangle$. Express $\langle E \rangle$ in terms of $\langle N \rangle$ and μ.
Hint: Ultra-relativistic means that you can neglect the mass term in the relativistic formula for the energy of the particles: $\epsilon(\boldsymbol{p}) = \sqrt{m^2c^4 + \boldsymbol{p}^2c^2} \approx pc$.

3 Relativistic Bose Gas [10pt]

Consider a relativistic Bose gas in 2D with dispersion relation $\epsilon(\vec{k}) = \sqrt{m^2c^4 + \vec{k}^2c^2}$.

1. Calculate the one-particle density of states.
2. Find expressions for N and E.
3. Should one expect Bose-Einstein condensation?

4 Relativistic Fermi Gas [10pt]

Consider a relativistic Fermi gas in 3D with dispersion relation $\epsilon(\boldsymbol{p}) = \sqrt{m^2c^4 + \boldsymbol{p}^2c^2}$.

1. Calculate the one-particle density of states.
2. Derive the following expressions for the thermodynamic properties of a relativistic Fermi gas

$$n = \frac{\langle N \rangle}{V} = \int_{mc^2}^{\infty} \frac{g(\varepsilon)}{z^{-1}\exp(\beta\varepsilon) + 1} d\varepsilon = \frac{g_s}{2\pi^2 c^3} \int_{mc^2}^{\infty} \frac{\sqrt{\varepsilon^2 - m^2c^4}}{\exp(\beta(\varepsilon - \mu)) + 1} \varepsilon d\varepsilon \tag{1}$$

$$\frac{E}{V} = \int_{mc^2}^{\infty} \frac{g(\varepsilon)}{z^{-1}\exp(\beta\varepsilon) + 1} \varepsilon d\varepsilon = \frac{g_s}{2\pi^2 c^3} \int_{mc^2}^{\infty} \frac{\sqrt{\varepsilon^2 - m^2c^4}}{\exp(\beta(\varepsilon - \mu)) + 1} \varepsilon^2 d\varepsilon \tag{2}$$

$$p = \frac{1}{3} \frac{g_s}{2\pi^2 c^3} \int_{mc^2}^{\infty} \frac{\sqrt{\varepsilon^2 - m^2c^4}^3}{\exp(\beta(\varepsilon - \mu)) + 1} d\varepsilon \tag{3}$$