Exercise sheet N. 7
Statistical Physics

University of Heidelberg

http://www.thphys.uni-heidelberg.de/~amendola/teaching.html

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Indicate your name, your UNI-ID and your exercise group on each page of the answer sheets.

Please hand in your solution on Tuesday 10th December during the lecture.

1 Equivalence of the Canonical and Macro-Canonical Ensembles

Consider the definition of partition function in the macro-canonical ensemble

$$\mathcal{Z}(T,V,\mu) = \sum_N e^{\beta\mu N} Z(T,V,N) \tag{1}$$

where $Z(T,V,N)$ is the partition function of the canonical ensemble for N indistinguishable particles, which can be written as

$$Z(T,V,N) = \frac{1}{N! h^{3N}} \int d^{3N}p\, d^{3N}q e^{-\beta H(q,p)}. \tag{2}$$

Given any observable θ we indicate its variance within the ensemble with the notation σ_θ. The average taken over the macro-canonical ensemble are indicated by the mc subscript or superscript, the ones taken over the canonical ensembles by the c subscript or superscript.

1. Write an expression for the energy fluctuations of a system in the macro-canonical ensemble $\Delta U_{mc}^2 = \frac{(\sigma_U^{mc})^2}{U^2} = \frac{\langle H^2\rangle_{mc} - \langle H\rangle_{mc}^2}{\langle H\rangle_{mc}^2}$.
2. Prove that for an ideal gas the energy variance in the macro-canonical ensemble can be related to its value in the canonical ensemble via the relation

$$\frac{(\sigma_U^{mc})^2}{U^2} = \frac{(\sigma_U^c)^2}{U^2} + \frac{\sigma_N^2}{U^2}\left(\frac{\partial U}{\partial N}\right)^2$$

   where $U = \langle H\rangle_{mc}$ is the average energy.
3. Assuming that the density of the gas is fixed $\rho = N/V$, prove that the gran-canonical ensemble is equivalent to the canonical one.

2 Particle Distribution of an Ideal Gas

Consider an ideal gas at temperature T in a finite box of volume V, the system is in contact with an infinite bath with which it can exchange both energy and particles. If the system is in thermodynamical equilibrium with the bath, the particle number in the volume V is expected to fluctuate around its mean value $\langle N\rangle = \bar{N}$. This configuration may be conveniently describe by the macro-canonical ensemble.

1. Write down the macro-canonical partition function for the system.
2. Compute the macro-canonical potential $\phi(T, V, \mu) = -PV$.
3. Compute the average particle number for the system $\bar{N} = \langle N \rangle$.
4. Compute the ideal gas chemical potential μ.
5. Prove that the probability of find N particles in the volume V is Poissonian.