Exercise sheet N. 5
Statistical Physics

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http://www.thphys.uni-heidelberg.de/~amendola/teaching.html

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Due to the Klimastreik week, the following exercise sheet will not be taken into account for the access to the final exam. Therefore, you should not hand in any solutions on Tuesday 3rd December. In any case, the exercises will be corrected during the tutorial groups in the dates 5th-6th December.

1 The fully connected Ising model

Consider a ring of N sites, which hosts N discrete spin variables σ_i, one per each site $i \in \{1, \cdots, N\}$. The spin variables can only assume two values $\sigma_i \in \{-1, 1\}$ and interact with each other by a ferromagnetic coupling term of strength J/N, where $J > 0$. The Hamiltonian of the system, considering also the effect of an homogeneous magnetic field h, reads

$$\mathcal{H} = -\frac{J}{N}\sum_{ij}\sigma_i\sigma_j - h\sum_i \sigma_i. \tag{1}$$

where $h > 0$ and the summation $\sum_{ij}$ run over all possible spin pairs, as we are considering a fully connected model, where interactions do not depend on the distance. Compute the free energy of the system in the thermodynamics limit ($N \to \infty$) and derive a closed expression for the average magnetisation

$$\bar{m} = \frac{\langle \sum_i \sigma_i \rangle}{N} \tag{2}$$

as a function of the temperature T and show that the system undergoes a transition from a state with $\lim_{h\to 0} m \to 0$ for $T > T_c$ and a state where $\lim_{h\to 0} m = 0$ and a state $\lim_{h\to 0} m \neq 0$ for $T < T_c$. Moreover, compute the value of T_c and demonstrate that $m \propto (T - T_c)^\nu$ with $\nu = 1/2$ for $T \approx T_c$.

Hint: in order to solve the exercise it is convenient to rewrite the Hamiltonian in terms of the global magnetisation

$$m = \sum_{i=1}^{N} \sigma_i / N.$$

Then, in the $N \to \infty$ limit it is possible to rewrite the total partition function of the system as an integral over the continuous magnetisation value $m \in [-1, 1]$ as

$$Z(T, N, h) = \int Z_m(T, N, h)dm.$$

The integral has the form

$$I(A) = \int_{x_1}^{x_2} f(x)e^{Ag(x)}, \tag{3}$$

in the large limit $A \gg 1$ the integral is dominated by the contributions around the maximum of $g(x)$. Therefore, assuming that $g(x)$ has a maximum at $x_1 < x_0 < x_2$, one can define a new integration variable y according to

$$x = x_0 + \frac{y}{\sqrt{A}}. \tag{4}$$

Expanding both f and g around x_0 one obtains the formula

$$I(A) = \frac{f(x_0)e^{Ag(x_0)}}{\sqrt{A}} \int_{y_1}^{y_2} dy e^{-y^2 g''(x_0)/2} \left(1 + \sum_{n=1}^{\infty} A^{-n/2} P_n(y)\right) \tag{5}$$

where P_n are polynomial functions. Since the integrand is very well peaked around $y = 0$, in the very large A limit one can neglects all the $O(A)$ contributions and extend the integration limits to infinity, yielding

$$I(A) \approx \sqrt{-\frac{2\pi}{Ag''(x_0)}} f(x_0)e^{Ag(x_0)} \tag{6}$$

using the above formula you can obtain the free energy of the model and answer to the questions proposed.