Exercise sheet N. 3
Statistical Physics

University of Heidelberg

http://www.thphys.uni-heidelberg.de/~amendola/teaching.html

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Indicate your name, your UNI-ID and your exercise group on each page of the answer sheets.

Please hand in your solution on Tuesday 12th November during the lecture.

1 Legendre Transforms [5pt]

Given the following functions

- $f(q) = e^q$,
- $f(q) = Aq^2$,
- $f(q) = \log(A + e^q)$,

where A is a real constant and $q \in \mathbb{R}$. For each function

(a) Specify the conditions under which the Legendre transform $\hat{f}(p)$ is well defined.

(b) Compute the Legendre transforms.

Do the domains of validity of each function and its Legendre transform agree?

2 The Hamiltonian of a relativistic particle [10pt]

The motion of a relativistic particle is described by the Hamiltonian

$$\mathcal{H}(q, \dot{q}) = \sqrt{p^2c^2 + m^2c^4} + U(q) \tag{1}$$

where q is the particle position and p is the particle momentum. $U(q)$ is the potential energy which depends only on the particle position, while m is the mass of the particle and c the light velocity.

(a) Define the particle momentum p and derive the system Lagrangian $\mathcal{L}(q, \dot{q})$.

(b) Given the relation $\mathcal{H} = K + U$, where K is the kinetic energy check if the relation $\mathcal{L} = K - U$ is obeyed.

(c) Is the Euler-Leagrange equation valid?

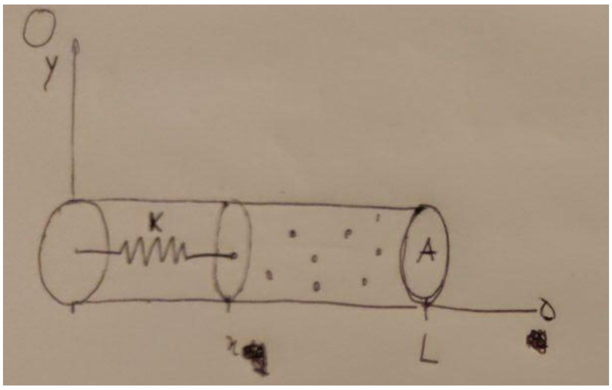

Figure 1: The system in exercise 3.

3 Ideal gas connected to elastic piston[10pt]

Consider a cylinder of length L and basis area A, see Fig. 1. The cylinder is separated into two parts by a rigid membrane, whose position along the length of the cylinder can vary. The membrane is connected to the left end of the cylinder by a spring with elastic constant k and rest position $x_0 = L$, while the right side of the cylinder is filled by an ideal gas at temperature T, which is kept fixed.

- Compute the equilibrium postition of the membrane x_{eq}

Hint: You can define the equilibrium by the condition $dF = 0$, where F is the Helmholtz free energy.

4 Binomial distribution [10pt]

Given an experiment with boolean outcome success/failure, the probability to have a successful outcome for the experiment is p (while the failure outcome as a probability $q = 1 - p$). After performing N independent trials of the same experiment the probability to have n successful outcomes is described by the Binomial distribution

$$P(n; N, p) = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}. \tag{2}$$

(a) What is the average number of successful trials $\langle n \rangle$.

(b) What is the variance of the number of successful trials.

For a very large number of experiments $N \gg n$ with very small success probability $p \sim 0$, show that the number of successful trial can be described by the Poissonian distribution

$$P(n; \nu) = \frac{\nu^n}{n!} e^{-\nu} \tag{3}$$

where $\nu = Np$.

5 Moments of a probability distribution [5pt]

Consider the following probability density function for the continuous variable x

$$f(x) = \frac{A}{x^2 + \gamma^2}. \tag{4}$$

Compute the value of A such that the probability density function is normalized and find the median of the distribution. Comment of the value of the expectation value of the variable x and on the higher moments of the distribution $f(x)$.